# Who Ya Gonna Call?

## A Solution to the Horse Race Problem with Application to Trade of Most Kinds

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# Overview

# Trade

Trade can sometimes be modeled as sealed bid auctions

1. Customer inquiry is sent to multiple dealers
2. Dealers compete on price
3. Trade is filled with best response

Of relevance to rare art, used cars and many financial securities.

# The dealer's problem

See Cotton and Papanicolaou [1] for proofs and more results.

## Hazard rate of the inside market

If $F(q)$ is the distribution of the most competitive markup from another dealer, and $f(q)$ the corresponding density function, we let $h(q) = \frac{f(q)}{1-F(q)}$ denote the corresponding hazard rate.

This is the sum of the hazards of the dealers' markup distributions. As a quick reminder of why let $F_i(z)$ and $f_i(z)$ denote the respective distributions and densities for the markups of the other dealers' markups.

# Who Ya Gonna Call?

Since the probability of winning the trade with a markup of $z$ is $1 - F(z) = \Pi_i(1 - F_i(z))$ we have, differentiating,

$$\begin{aligned} -f(z) &= \sum_i -f_i(z)\Pi_{j\neq i}(1 - F_j(z)) \\ &= -\sum_i \frac{f_i(z)}{1 - F_i(z)}\Pi_j(1 - F_j(z)) \\ &= -(1 - F(z))\sum_i \frac{f_i(z)}{1 - F_i(z)} \end{aligned}$$

so

$$\frac{f(z)}{1 - F(z)} = \overbrace{\sum_i \frac{f_i(z)}{1 - F_i(z)}}^{sum\ of\ hazards}$$

## Optimal markups and the inside hazard

The optimal markups can be expressed in relation to $K^{\downarrow}(x;s)$ representing a non-myopic indifference markup pertaining to the optimal strategy. They must relate to the hazard rate of the inside market.

$$\overbrace{\left(m^{ask} - K^{\downarrow}(x;s)\right)}^{existing\ benefit} \quad \overbrace{\underbrace{h(m^{ask})}_{inside\ hazard} \quad \Delta m}^{chance\ of\ losing\ it} = \overbrace{\Delta m}^{increase\ in\ profit} \qquad .$$

The obvious increase in profit (RHS) when increasing the markup by $\Delta m$ exactly offsets the downside. The downside arises from the possibility of a competitor's ask falling right between $m$ and $m + \Delta m$ - the probability of that is proportional to the hazard rate of the inside market. In this event the upside of the trade at markup $m$ which would otherwise have been won will be forfeited.

# Who Ya Gonna Call?

The indifference markups $K$ relate to indifference inventory cost $\nu()$, also defined with respect to the optimal strategy. In the paper the indifference markups are defined in terms of $\nu()$ via:

$$K^{\uparrow}(x;s) \quad := \quad \epsilon + \frac{\nu(x+s) - \nu(x)}{s} \tag{1}$$

when marking down and similarly

$$K^{\downarrow}(x;s) \quad := \quad \epsilon + \frac{\nu(x-s) - \nu(x)}{s} \tag{2}$$

when marking up. The interpretation of $K$ follows after the fact.

Here $\epsilon$ is a constant accounting for adverse selection.

The optimal dealer strategy is to markup by

$$m^{ask}(x;s) = \frac{1}{h(m^{ask}(s))} + K^{\downarrow}(x;s)$$

which is a simple re-arrangement of our observation about the inside markup hazard rate. It leaves open the properties of $\nu()$ buried in the $K$'s.

As an example if the inside hazard is a constant $h$ corresponding to exponentially distributed inside markup with mean $w = 1/h$, which we interpret as a market width, then the optimal policy is

$$m^{ask}(x;s) = w + \epsilon + \frac{\nu(x-s) - \nu(x)}{s} \tag{3}$$

In words:

markup = markup width + adverse selection + marginal inventory cost

# Who Ya Gonna Call?

The dealer weights direct carrying costs against the fact that holding more inventory makes the next trade opportunity more valuable to her.

$$
\begin{aligned}
\frac{\overbrace{\tau c(x)}^{storage\ cost}}{s} = \; & p \sup_{m^{\uparrow}} \left\{ \overbrace{\left(m^{\uparrow} - K^{\uparrow}(x;s)\right)}^{net\ gain} \overbrace{P^{\uparrow}(m^{\uparrow})}^{fill\ probability} \right\} \\
& + (1-p) \sup_{m^{\downarrow}} \left\{ \left(m^{\downarrow} - K^{\downarrow}(x;s)\right) P^{\downarrow}(m^{\downarrow}) \right\} \\
& - p \sup_{m^{\uparrow}} \left\{ \left(m^{\uparrow} - K^{\uparrow}(0;s)\right) P^{\uparrow}(m^{\uparrow}) \right\} \\
& - (1-p) \sup_{m^{\downarrow}} \left\{ \left(m^{\downarrow} - K^{\downarrow}(0;s)\right) P^{\downarrow}(m^{\downarrow}) \right\}
\end{aligned}
$$

where $p$ is the probability that a client wants to sell (versus buy). This is a Bellman equation with a value function $-\nu$ buried inside the break-even markups $K$.

## Order imbalance

If there are more buying opportunities for our market maker than selling opportunities, or vice versa, we will have the analytic inconvenience $p \neq \frac{1}{2}$. Define

$$\delta = \frac{\log(1-p) - \log(p)}{2h}$$

and

$$\gamma = \frac{\log\left(\frac{1}{2}\right) - \frac{\log(1-p)+\log(p)}{2}}{h}$$

It is clear from concavity of $\log$ that $\gamma > 0$.

Claim: Order imbalance requires the market maker to skew by $\delta$ and widen by $\gamma$.

Claim: Order imbalance increases effective carrying cost by

$$e^{\log\left(\frac{1}{2}\right) - \frac{\log(1-p)+\log(p)}{2}} > 1$$

...but enough about the dealer perspective.

## The customer's problem

Who to call? Everyone? What if there are costs?

1. Time and effort on customer's part
2. Time and effort on dealer's part
3. Sensitivity of customer inventory and intent

Customer needs to choose a set of dealers $V$ from the powerset $2^S$ of all dealers.

## Problem formulation

Given a desire to trade with size $s$ the buyer of an asset decides who to call by maximizing the following utility function:

$$V^*(s) = argmin_{V \in 2^S} \left\{ \overbrace{s\,E\left[\min_{i \in V} m_i\right]}^{markup} + \overbrace{I(V;s)}^{cost} \right\} \quad (4)$$

where $s$ is the size of the potential trade, $S = 1, \ldots, n$ is the set of all dealers, $V$ is a subset and $I(V;s)$ is the economic cost of inquiry for trade size $s$ to a group $V$ of dealers incorporating both informational and direct costs.

We will assume $I(V;x)$ is exogenous and, for simplicity, linear:

$$I(V;s) = \sum_{i \in V} sI_i$$

for some dealer parameters $I_i$ fixed in advance.

But how to model the expected markup for any subset of dealers in a consistent manner?

$$average\ markup = \overbrace{E\left[\min_{i \in V} m_i\right]}^{markup}$$

where $m_i$ is the stochastic markup offered by the $i$'th dealer.

## Market share

It seems unrealistic to expect clients to individually model the markup distributions of every dealer.

Instead we offer a simple characterization. We assume that if all dealers are asked then the unconditional probability of the $i$'th dealer providing the best price is $p_i$.[1]

Question: How to model the distribution of disseminated dealer prices in a manner consistent with market share?

[1] If no better method exists the quantity $p_i$ may inferred from market share. Such numbers are often publicized.

# The Horse Race Problem

The continuous horse race problem assumes a density $f()$ with distribution $F()$ and seeks offsets $(a_1, \ldots, a_n)$ in order to satisfy

$$p_i = \int_{-\infty}^{\infty} f^{\rightarrow a_i}(x) \Pi_{j \neq i}^{n} \left(1 - F^{\rightarrow a_j}(x)\right) dx$$

for some specified winning probabilities $p_i$. Here $f^{\rightarrow a_i}$ is the density translated by a constant $a_i$.

This is (arguably) the fundamental challenge for risk neutral pricing of equine derivatives such as trifectas, quinellas, exactas, pick four, superfectas, place and show.

Next ... we will define a morally equivalent problem for discrete distributions with just a little extra setup. The complexity arises from the possibility of dead heats.

## Harville: exponentially distributed performance

The case of $f()$ exponential is named for Harville [2] and can be solved in your head.

Editorial:

1. In one mental model consistent with Harville horses stay at the barrier after the gates open for an indefinite time. Then one at a time they awake and instantly traverse the entire course.

2. The onus falls on Harville fans to justify a more convincing mental model for horses (as distinct from radioactive particles).

## Henery: normally distributed performance

The case of normal $f()$ is considered by [4] [3] and approximate analytical results derived by Taylor expansion:

$$a_i = \frac{(n-1)\phi\left(\Phi^{-1}(\frac{1}{n})\right)\left(\Phi^{-1}(p_i) - \Phi^{-1}(\frac{1}{n})\right)}{\Phi^{-1}\left(\frac{i-\frac{3}{8}}{n+\frac{3}{4}}\right)}$$

where $\phi$ and $\Phi$ are the standard normal density and cumulative distribution respectively.

## More analytical possibilities ...

A theoretical comparison between Harville and Henery's approach is made in [5]. An ad-hoc attempt to improve Harville by replacing $p_i$ with $p_i$ raised to a power $\beta$ (then normalized across horses) is suggested. Other suggestions are made in [8] who previously noted the tractibility of the case of Gamma distributed $X_i$ in [7].

## Aside for baseball fans ... maybe hockey

The famous Bill James' Pythagorean formula

$$\overbrace{p}^{win\ probability} = \frac{\overbrace{RS^2}^{season\ runs}}{RS^2 + \underbrace{RA^2}_{season\ runs\ against}} \tag{5}$$

is precisely correct if runs scored in a game follows the Weibull distribution.[6]

Great for two horse races!

## Drawbacks with prior approaches

In our context we'll be looking to model the distribution of prices disseminated back to clients in response to an inquiry. These distributions reflect the economics and microstructure, which can vary widely, and

1. are unlikely to be exponential or gaussian;
2. may have thin tails;
3. may be supported on a lattice (quantized pricing);
4. may be highly asymmetrical...

And so forth

# The Discrete Horse Race Problem

Let $X_1, ..., X_n$ be discrete univariate contestant scoresassumed to take values on a lattice of equally spaced points. Let $X^{(k)}$ denote the $k$'th order statistic and in particular let $X^{(1)}$ denote the winning minimum score.

We define the implied state price for each contestant $i$ as follows.

$$p_i = E\left[\frac{\iota_{X_i = X^{(1)}}}{\sum_k \iota_{X_k = X^{(1)}}}\right] \tag{6}$$

where $\iota$ is the indicator function. The price $p_i$ is the expected payout in a game where we get

$$payoff = \begin{cases} 1 & \text{if horse i wins} \\ \frac{1}{2} & \text{if tied with one other} \\ \frac{1}{3} & \text{if tied with two others} \\ \dots & \end{cases}$$

It reduces to the probability of winning if there are no ties.

## Approximate translation operator

For any $f : \mathbb{N} \to \mathbb{R}$ and any $a \in \mathbb{R}$ we define the shifted distribution $f^{\to a}(\cdot)$.

$$f^{\to a} = (1 - r) f^{\to \lfloor a \rfloor} + r f^{\to \lfloor a \rfloor + 1} \tag{7}$$

where $r = a - \lfloor a \rfloor$ is the fractional part of the shift $a$ obtained by subtracting the floor. This operation takes a density of $X$ to one that approximates the density of $X + a$.

Remarks:

1. Exact translation if integer $a$
2. The same mean as $X + a$

## Formal definition of the Discrete Horse Race Problem

Given a distribution $f(\cdot)$ on the integers and a vector $\{p_i\}_{i=1}^n$ of state prices summing to unity, find a vector of offsets $(a_1, \dots, a_n)$ such that the following holds for every $i$ when the distribution of the $i$'th score $X_i$ is given by $f^{\to a_i}$.

$$p_i = E\left[\frac{\iota_{X_i = X^{(1)}}}{\sum_k \iota_{X_k = X^{(1)}}}\right]$$

Remarks:

1. Can set $a_0 = 0$ w.l.o.g.
2. Don't really need $f()$ common across all horses.
3. A "mere" optimization ... but try it for $n = 300$

## The method of multiplicity iteration

It will be convenient to define

$$S_i(j) = Prob(X_i > j) = 1 - F_i(j) \tag{8}$$

as the $i$'th survival function.

Define the (conditional) multiplicity to be the expected number of variables that tie for the lowest value, assuming the lowest value is precisely $j$:

$$m(j) = E\left[\sum_{i=1}^{n} \iota_{X_i=j} | X^{(1)} = j\right] \tag{9}$$

## Multiplicity calculus

Now suppose $X_1, ..., X_n$ represent the minimums of groups of (non-overlapping) variables, with respective multiplicities $m_1, ..., m_n$ respectively.

Take the union of the first two groups. The multiplicity is:

$$\begin{aligned} m_{1,2}^{(1)}(j) &= \frac{numer}{denom} \qquad (10) \\ numer &= m_1(j) f_1(j) S_2(j) + (m_1(j) \\ &\quad + m_2(j)) f_1(j) f_2(j) + m_2(j) f_2(j) S_1(j) \\ denom &= f_1(j) S_2(j) + f_1(j) f_2(j) + f_2(j) S_1(j) \end{aligned}$$

by careful accounting.

**Algorithm 1:** Density and multiplicity

**Input:** Discrete densities $f_i : \mathbb{N} \to \mathbb{R}$, multiplicities $m_i : \mathbb{N} \to \mathbb{R}$ for $i \in \{1, \ldots, n\}$ ;

Initialize $S = S_1$, $f = f_1$, $m = m_1$ ;

**for** $i = 2$ *to* $n$ **do**

- $S \to 1 - (1 - S)(1 - S_i)$;
- $f(j) = S(j) - S(j-1)$ for all $j$;
- Assign $m(j)$ for all $j$ using eqn 10 with $f, m, S$ taking the role of group 1 and $f_i, m_i, S_i$ the role of group 2.

**end**

## Iterating with one horse out

The idea of our algorithm is to estimate the "best of the rest" density assuming some given offsets $a_i$, then adjust, then re-estimate and so forth. After each iteration and choice of $a_i$ we compute the survival function $S$ and multiplicity $m$ for the set of all horses in the race.

## Pricing against the rest

Recalling the implied state price:

$$p_i = E\left[\frac{\iota_{X_i = X^{(1)}}}{\sum_k \iota_{X_k = X^{(1)}}}\right]$$

Compute by conditioning on the winning score $j$:

$$p_i = \sum_j \frac{f_{\hat{i}}(j) f_i(j)}{1 + m_{\hat{i}}(j)} \tag{11}$$

where $f_{\hat{i}}(j)$ and $m_{\hat{i}}$ are the density and multiplicity for the first order statistic in the complement of $\{j\}$ (i.e. all the competitors except horse $i$).

## Multiplicity inversion

The multiplicity calculus can be inverted to determine the multiplicity for all-but-one of the horses.

$$\begin{aligned} m_{\hat{i}}(j) &= \frac{numer}{denom} \\ numer &= m(j)f_1(j)S_{\hat{i}}(j) + m(j)f_1(j)f_{\hat{i}}(j) \\ &\quad +m(j)f_{\hat{i}}(j)S_1(j) - m_1(j)f_1(j)S_{\hat{i}}(j) \\ &\quad -m_1(j)f_1(j)f_{\hat{i}}(j) \\ denom &= f_{\hat{i}}(f_1 + S_1) \end{aligned} \tag{12}$$

**Algorithm 2:** Iterative Solution to Discrete Horse Race Problem

**input:** Win probability $p_i$, discrete density $f$ ;

Initialize $a_i = 0$ for all $i$. ;

**while** $\tilde{p}_i \not\approx p_i$ *for any* $i$ **do**

  Apply Algorithm 1 to compute $S, m, f$ from the collection of densities $f_i := f^{\rightarrow a_i}()$ ;

  **for** $i = 1$ ***to*** $n$ **do**

    Compute $S_{\hat{i}}$, $m_{\hat{i}}$ using eqn 12 ;

    Compute implied state prices $\tilde{p}_i$ for all $i$ using eqn 11 ;

  If any $\tilde{p}_i - p_i$ is too large assign new $a_i$ for all $i$ by assuming $a \rightarrow \tilde{p}$ is linear ;

**output:** $a_i$

## Summary: Horse Race Solution

We have exhibited a fast algorithm for inferring the location parameters of variables $X_i$ when partial information is available:

1. The probability[2] that $X_i$ is least among $X_1, \ldots, X_n$
2. The distribution of $X_i$ up to a translation.

In contrast to previous work the distribution $f()$ is not assumed to fall into a known family, but kept general.

[2] Technically the state price in the discrete case

# Solving the customer's problem

Armed with fast calibration we return to the customer problem.

$$V^*(s) =_{V \in 2^S} \left\{ s \, \overbrace{E\left[\min_{i \in V} m_i\right]}^{markup} + \overbrace{I(V; s)}^{cost} \right\}$$

We can now model the first term.

**Algorithm 3:** Who to call

**Input:** Dealer market share probability $p_i$, discrete density $f$ for dealer markup, information and search costs $I_i$ for $i \in \{1, \ldots, n\}$ ;

Determine $a_i$ using Algorithm 2 ;

Choose subset $V$ of dealers to minimize

$$\Psi(V) := s\, \overbrace{E\left[\min_{i \in V} m_i\right]}^{\phi(V)} + \overbrace{I(V;s)}^{cost}$$

**Output:** Subset $V$ of $\{1, \ldots, n\}$

## Submodularity

Recall that a function $g$ from $2^n \to \mathbb{R}$ is submodular if for any two subsets $V \subseteq W \subseteq S = \{1, \dots, n\}$ we have

$$g(V \cup W) + g(W \cap V) \leq g(V) + g(W) \tag{13}$$

Why do we care? Minimizing submodular functions is easy and we can prove that the customer's task falls in this category.

We claim that

$$\phi(V) = E\left[\min_{i \in V} m_i\right]$$

is submodular from which it would follow, by linearity of $I(V; x)$ that inequality 13 also holds for $\Psi(V)$, the quantity we the customer wishes to minimize.

## Diminishing returns

To prove that $\phi()$ is submodular we turn to an equivalent definition of submodularity. Denote by $\phi(A, i) := \phi(A + i) - \phi(A)$ the marginal value of $i$ with respect to A. This is the expected gain to be made in the price by calling one additional dealer $i$.

It is well known that $\phi$ is submodular if there are decreasing returns. That is to say that for all $V \subseteq W \subseteq \{1, \ldots, n\}$ and $i \in W$, $\phi(V, i) \geq \phi(W, i)$

It is intuitive that entering a horse in a small field will reduce the average winning time by more than if the same horse is entered in a larger field which contains all the horses as the first race and some more.

## Proof of diminishing returns

We introduce notation

$$m_W := \min_{j \in W} m_j$$

for the stochastic variable representing the minimal markdown over any set of dealers.

Observe that as we add a horse the following quantity

$$\begin{aligned} \delta_W(i) &= \min_{j \in W} m_j - \min_{j \in W \cup \{i\}} m_j \\ &= \left\{ \begin{array}{cc} 0 & m_i \geq m_W \\ m_W - m_i & m_i < m_W \end{array} \right. \\ &\geq 0 \end{aligned}$$

is always positive.

Then for $V \subset W$ we have $m_V \geq m_W$ and thus

$$\delta_W(i) - \delta_V(i) = \begin{cases} 0 & m_i \geq m_V \geq m_W \\ m_i - m_V & m_W \leq m_i < m_V \\ m_W - m_V & m_i < m_W \leq m_V \end{cases}$$

In each case $\delta_W(i) - \delta_V(i) \leq 0$.

Since both $\delta_W(i)$ and $\delta_V(i)$ are non-negative we can define a quantity $\eta(W, V)$ such that

$$\eta(W, V)\delta_V = \delta_W$$

with $0 \leq \eta(W, V) \leq 1$.

Diminishing returns for$\phi(\cdot)$ now follows by iterated expectations, taking advantage of the fact that dealer bids are independent.

$$
\begin{aligned}
\phi(W,i) &= E\left[\min_{j\in W} m_j\right] - E\left[\min_{j\in W\cup\{i\}} m_j\right] \\
&= E_V\left[E_{W\setminus V}\underbrace{\left[\min_{j\in W} m_j - \min_{j\in W\cup\{i\}} m_j\right]}_{=:\delta_W(i)}\right] \\
&= E_V\left[E_V\left[\delta_V(i)E_{W\setminus V}\underbrace{\left[\eta(W,V)\right]}_{\leq 1}\right]\right] \\
&\leq E_V\left[\delta_V(i)\right] \\
&= \phi(V,i)
\end{aligned}
$$

We have proven that $\phi(V)$ and therefore $\Psi(V)$ are submodular.

# Summary

We have provided an algorithm that chooses a subset of dealers for a customer to call, where the only required inputs are:

1. The cost of calling dealer $i$ per unit trade size
2. The probability that dealer $i$ would offer the best price if we were to call all dealers.

This applies to the general case where the distribution of disseminated prices can be arbitrary. The objective is submodular and thus an "easy" minimization.

In the process we have provided a numerical solution to the pricing of all possible exotic horse race wagers.

**Algorithm 4:** Memoized multiplicity

**input:** Discrete densities $\{f_i\}_{i=1}^{n}$. ;
**input:** Sets $V_k \subset \{1, \ldots, n\}$ for $k \in K$ with previously computed multiplicities, densities and survivals $\Upsilon(V_k) = (m, f, S)$. ;
**convention** $V_0$ is empty set. ;
**parameter:** $\alpha \in (0, 1)$ ;
Choose $k^* =_k \{\alpha|V \setminus V_k| + |V_k \setminus V|\}$ ;
**if** $k^* = 0$ **then**
  Fallback to using Algorithm 1 ;
Assign $W = V_{k^*}$ ;
**for** $i \in W \setminus V$ **do**
  Compute $\Upsilon(W \setminus \{i\})$ from $\Upsilon(W)$ ;
  Set $W = W \setminus \{i\}$ ;
Compute $\Upsilon\left(W^{\complement}\right)$ on complement in $V$;
Compute $\Upsilon(V)$ from $\Upsilon\left(W^{\complement}\right)$ and $\Upsilon(W)$

# References

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